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FAST TRACK COMMUNICATION

Symplectic rectification and isochronous Hamiltonian systems

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Abstract
We report the connection of symplectic rectification in the construction of isochronous Hamiltonian systems.

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1. Introduction

Following the recent works of Calogero and his coworkers [2, 3] on isochronous Hamiltonian systems, we show how a large number of planar isochronous systems can be constructed in a systematic way. The beauty of Calogero's formalism lies in its inherent simplicity for it essentially employs the Hamiltonian of the linear harmonic oscillator, albeit in a novel manner. Undoubtedly, this is an interesting scheme which yields some surprising results. In a series of recent papers, Calogero and Levyraz [4, 5] have described a new technique for deducing isochronous Hamiltonian systems. Essentially, it involves transforming a real autonomous Hamiltonian $H(q, p)$ into a 'Ω'-modified Hamiltonian such that the dynamics is now isochronous. The procedure requires the introduction of a function, $\Theta(q, p)$, which behaves like a collective coordinate conjugate to the Hamiltonian, such that the Poisson bracket $\{\Theta, H\} = 1$. The modified Hamiltonian has the following appearance: $\widetilde{H} = \frac{1}{2}(H^2 + \Omega^2\Theta^2)$. Here, Ω is an arbitrary positive constant and it is evident that H plays the role of the new momentum. With this particular form of the modified Hamiltonian, H and Θ evolve sinusoidally with time and period $T = 2\pi/\Omega$. By inverting H and Ω, one can obtain expressions for q and p respectively. Since H and Θ evolve sinusoidally, p and q must necessarily evolve with the same period $T = 2\pi/\Omega$.

Rectification of the vector field [1] is a diffeomorphism that transforms it into a field of parallel vectors of identical length in the Euclidean space. In general, every differential equation $\dot{\mathbf{x}} = \mathbf{W}(\mathbf{x})$ can be written in the normal form $\dot{x_1} = 1, \dot{x_2} = \cdots = \dot{x_n} = 0$ for a suitable choice of rectifying coordinates in a sufficiently small neighbourhood of any nonsingular point of the field. In other words, every equation $\dot{\mathbf{x}} = \mathbf{W}(\mathbf{x})$ is locally equivalent to the simplest equation in a neighbourhood of any nonsingular point.

We consider a special type of rectification associated with flow in the phase space $\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x})$, where $\mathbf{G}(\mathbf{x}) = \left(\left\{\frac{\partial H}{\partial p_i}\right\}, -\left\{\frac{\partial H}{\partial q_i}\right\}\right)$. The symplectic structure of the canonical equation has two consequences: (a) to solve the autonomous system of $2n$ equations, n integrals are sufficient and (b) the size of a volume in the phase space remains constant as it flows.

In this brief communication, we describe how by symplectic rectification [6] of the Hamiltonian H, which plays an auxiliary role, one can identify the collective coordinate Θ and thereby construct a modified Hamiltonian $\widetilde{H}$ exhibiting isochronicity.

2. Symplectic rectification

We begin by briefly recollecting some basic ideas regarding symplectic rectification of Hamiltonian systems. Consider a first-order autonomous system of differential equations

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{X}(\mathbf{x}), \tag{2.1}$$

where $\mathbf{x} = (x^1, x^2, \ldots, x^{2n}) \in \mathbb{R}^{2n}$. The associated vector field X in $\mathbb{R}^{2n}$ is a Hamiltonian if there exists a function $H(x)$ in C^2 such that

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = J\nabla_{\mathbf{x}} H(\mathbf{x}), \tag{2.2}$$

where $J = \left(\begin{smallmatrix} \mathbb{O} & \mathbb{I} \\ -\mathbb{I} & \mathbb{O} \end{smallmatrix}\right)$ is a real $2n \times 2n$ orthogonal skew-symmetric matrix and $\nabla_{\mathbf{x}} H$ denotes the symplectic gradient of the Hamiltonian H. Note that $\mathbb{O}$ and $\mathbb{I}$ represent $n \times n$ null and identity matrices respectively. Under these circumstances, one says that the system of equation (2.1) is a Hamiltonian. The problem of integrating such a system is often greatly simplified by making an appropriate change of variables such that the resulting system is easier to solve. Canonical transformations are a class of point transformations and preserve the canonical structure (2.2) of Hamilton's equations of motion. This means that if $\mathbf{x} = (\mathbf{q}, \mathbf{p}) \in R^{2n}$ such that $\dot{\mathbf{x}} = J\nabla_{\mathbf{x}} H(\mathbf{x})$, then under a diffeomorphic coordinate transformation $\mathbf{x} = \mathbf{x}(\mathbf{X})$ with $\mathbf{X} = (\mathbf{Q}, \mathbf{P}) \in R^{2n}$ the system of equations is transformed to

$$\dot{\mathbf{X}} = \mathbf{W}(\mathbf{X}) = J\nabla_{\mathbf{X}} K, \tag{2.3}$$

where $K(\mathbf{X})$ is the new Hamiltonian being expressed as a function of the variables $(\mathbf{Q}, \mathbf{P})$.

The technique of symplectic rectification, which we shortly illustrate, leads to a pair of conjugate variables taking the values of the Hamiltonian (constant) and of time, while for $n > 1$ the remaining coordinates of the phase space are *all* first integrals of motion. Restricting the discussion to $n = 1$ dimensional systems, the objective is to determine a *completely* canonical transformation of the variables (q, p) such that the new coordinate

$$P = H \qquad \text{and} \qquad Q(q, p) = t - t_0. \tag{2.4}$$

This follows from the observation that for a completely canonical transformation, the new Hamiltonian $K(Q, P) = P$ so that the canonical equations are $\dot{P} = 0$ and $\dot{Q} = 1$. The procedure is based on the following theorem.

Theorem 2.1 (Symplectic rectification)**.** *Let $H(x)$ be a C^1-function in $\mathbb{R}^{2n}$ and $x_0 \in \mathbb{R}^{2n}$ such that $\nabla_x H(x_0) \neq 0$. Then there exists a completely canonical transformation $X = X(x)$, defined in a neighbourhood of x_0, such that $K(X) = H(x(X)) = X_i$ for some i.*

Its proof is given in [6]. For a one-dimensional Hamiltonian system with $\dot{q} = \frac{\partial H}{\partial p}$ and $\dot{p} = -\frac{\partial H}{\partial q}$, suppose the initial conditions are $q(0) = \xi$ and $p(0) = \eta$ respectively. Next, consider the retrograde flow defined by the Hamiltonian $f(\xi, \eta) = -H(\xi, \eta)$ with initial conditions $\xi(0) = q$ and $\eta(0) = p$. By hypothesis, we may assume $\frac{\partial H}{\partial p} \neq 0$ which in turn implies $\dot{\xi} = \frac{\partial f}{\partial \eta} = -\frac{\partial H}{\partial \eta} \neq 0$ for the retrograde flow. Let $\phi(\xi, \eta) = 0$ denote a regular curve in the (ξ, η) plane. Assume that $\phi(\xi, \eta) = 0$ is not a trajectory of the flow determined by $f(\xi, \eta)$. Considering a point (q, p) such that the trajectory under f intersects the curve $\phi(\xi, \eta) = 0$ at time $t = \bar{t}(q, p)$, it may be shown that $\bar{t}$ can be determined from the equation

$$\phi(\xi(q, p, \bar{t}),\ \eta(q, p.\bar{t})) = 0. \tag{2.5}$$

The determination of $\bar{t}$ allows us to complete the canonical transformation $(q, p) \longrightarrow (Q, P)$ through the identification

$$P = H(q, p) = \text{constant}, \qquad Q(q, p) = \bar{t}(q, p). \tag{2.6}$$

The canonical nature may be explicitly verified by evaluating the Poisson bracket $\{Q, P\} = \{\bar{t}(q, p), H(q, p)\} = 1$.

2.1. Illustration of symplectic rectification via examples

In this section, we illustrate the technique by considering the following example of Calogero and Levyraz in [5].

Example 1. $H = wpq$.

We use symplectic rectification to determine the collective coordinate function $\Theta(q, p)$ such that $\{\Theta, H\} = 1$. With the standard Poisson bracket $\{q, p\} = 1$, the equations of motion are $\dot{q} = wq$ and $\dot{p} = -wp$. Their solutions are $q(t) = \xi\, \mathrm{e}^{wt}$ and $p(t) = \eta\, \mathrm{e}^{-wt}$, respectively. Here, $\xi = q(0)$ and $\eta = p(0)$ represent the initial conditions. The retrograde Hamiltonian is $f = -w\xi\eta$ with $\{\xi, \eta\} = 1$. The solutions of the canonical equations of motion for the retrograde Hamiltonian with initial conditions $\xi(0) = q$ and $\eta(0) = p$ are given by

$$\xi(t) = q\, \mathrm{e}^{-wt} \qquad \text{and} \qquad \eta(t) = p\, \mathrm{e}^{wt}, \tag{2.7}$$

respectively. By the conditions of the previous theorem, there exists a canonical transformation such that the new Hamiltonian $K(Q, P) = P$ which implies by virtue of the equations of motion $P =$ constant, so that we may set $P = H = wpq$, while the new coordinate is $Q = t - t_0$. We consider the regular curve

$$\phi(\xi, \eta) = \xi - a = 0, \tag{2.8}$$

where a is an arbitrary constant. It is easily checked that $\{\phi(\xi, \eta), f\} = \{\xi - a, -w\xi\eta\} = -w\xi \neq 0$ and so $\xi = a$ is not a trajectory under f. Fixing a point (q, p) such that the trajectory (2.7) intersects the curve (2.8) at $t = \bar{t}$, we find that $\bar{t} = \frac{1}{w}\log\left(\frac{q}{a}\right)$. Then, as $Q(q, p) = t - t_0$, setting $t_0 = 0$ we obtain $Q = \bar{t}(q, p) = \frac{1}{w}\log\left(\frac{q}{a}\right)$. It is easy to check that $\{Q, P\} = 1$, thereby verifying the canonical nature of the transformation:

$$P = wpq, \qquad Q = \frac{1}{w}\log\left(\frac{q}{a}\right). \tag{2.9}$$

The latter is precisely the expression for the collective coordinate Θ in the example in [5].

In the following example, we show how using symplectic rectification we may derive isochronous Hamiltonian systems following Calogero's method.

Example 2. $H = q\cot p$

In this case, Hamilton's equations are $\dot{q} = -\frac{q}{\sin^2 p}$ and $\dot{p} = -\cot p$. Their solutions are given by

$$q(t) = \frac{\xi}{\sin\eta}\sqrt{\mathrm{e}^{-2t} - \cos^2\eta}, \qquad p(t) = \arccos(e^t\cos\eta), \tag{2.10}$$

where $q(0) = \xi$ and $p(0) = \eta$. Note that the inverse trigonometric functions are multivalued, so we consider their principal values.

For the flow determined by the retrograde Hamiltonian $f(\xi,\eta) = -\xi\cot\eta$ under the initial conditions $\xi(0) = q$, $\eta(0) = p$, we find that

$$\xi(t) = \frac{q}{\sin p}\sqrt{\mathrm{e}^{2t} - \cos^2 p}, \qquad \eta(t) = \operatorname{arccot}(\mathrm{e}^{-t}\cos p). \tag{2.11}$$

The choice of the curve $\phi(\xi,\eta)$ is not unique and different choices yield different canonical transformations. However, it has to be ensured that the curve ϕ is not a trajectory. Suppose we take

$$\phi(\xi,\eta) = \frac{\xi}{\sin\eta} = \alpha, \tag{2.12}$$

where α is an arbitrary constant. As explained in the previous example, for fixed (q, p) on the trajectory (2.11), its intersection with the curve (2.12) occurs at time $t = \bar{t}$ where

$$\bar{t}(q, p) = \log\left(\frac{\sin p}{q}\right).$$

Hence, we arrive at the following canonical transformation:

$$P(q, p) = q\cot p \qquad \text{and} \qquad Q(q, p) = \log\left(\frac{\sin p}{q}\right) \tag{2.13}$$

with $\{q, p\} = 1$. We then define the Ω-modified Hamiltonian by

$$\widetilde{H} = \frac{C}{2}\left[\left(\frac{q\cot p}{C}\right)^2 + \Omega^2\log^2\left(\frac{\sin p}{q}\right)\right]. \tag{2.14}$$

The constant C has been introduced purely to ensure dimensional consistency. By construction, $\{Q, P\} = 1$ and (2.14) is just the harmonic oscillator Hamiltonian. The time evolution of P and Q is given by

$$\dot{Q} = \{Q, \widetilde{H}\} = \frac{P}{C} \qquad \dot{P} = \{P, \widetilde{H}\} = -C\Omega^2 Q. \tag{2.15}$$

These have solutions

$$\begin{aligned} Q(t) &= Q(0)\cos(\Omega t) + \frac{P(0)}{C\Omega}\sin(\Omega t) \\ P(t) &= P(0)\cos(\Omega t) - CQ(0)\Omega\sin(\Omega t) \end{aligned} \tag{2.16}$$

and evolve periodically with period $T = \frac{2\pi}{\Omega}$. From (2.13), it follows that

$$\cos p = Pe^Q \quad \text{so that} \quad p(t) = \cos^{-1}[Pe^Q] \tag{2.17}$$

and

$$q = \mathrm{e}^{-Q}\sin p = \mathrm{e}^{-Q}\sin[\cos^{-1}(Pe^Q)]. \tag{2.18}$$

In view of the sinusoidal dependence of P and Q on t as evident from (2.16), we conclude that q and p evolve with the same period $T = \frac{2\pi}{\Omega}$.

3. A Liénard-type equation

In this section, we investigate the applicability of the technique described above to ordinary differential equations of the following class:

$$\ddot{x} + F(x)\dot{x}^2 + G(x) = 0. \tag{3.1}$$

The isochronicity problem and the properties of the periodic function were studied by Sabatini [7]. Let us express (3.1):

$$\dot{x} = y, \qquad \dot{y} = -G(x) - F(x)y^2 \tag{3.2}$$

and denote $f(x) = \int_0^x F(s)\,\mathrm{d}s$ and $\phi(x) = \int_0^x \mathrm{e}^{f(x)}\,\mathrm{d}s$. It was demonstrated in [7] that by substitution $u = \phi(x)$, equation (3.2) can be transformed into the system

$$\dot{u} = y, \qquad \dot{y} = -G(\phi^{-1}(u))\,\mathrm{e}^{f(\phi^{-1}(u))}.$$

This system is a particular case of the system

$$\dot{x} = y, \qquad \dot{y} = -z(x). \tag{3.3}$$

Denote $U(x) = \int_0^x z(s)\,\mathrm{d}s$; then the first integral is $H(x, y) := \frac{y^2}{2} + U(x) = E$, where $H(x, y)$ is the Hamiltonian of (3.2).

Consider the following system of first-order ordinary differential equations:

$$\dot{x} = f(x)y, \qquad \dot{y} = -\frac{f'(x)}{2}y^2 + \Omega^2 h(x). \tag{3.4}$$

This is equivalent to the following second-order equation:

$$\ddot{x} = \frac{1}{2}\frac{f'(x)}{f(x)}\dot{x}^2 + \Omega^2 f(x)h(x), \tag{3.5}$$

so comparison with (3.1) shows that $F(x) = -\frac{1}{2}\frac{f'(x)}{f(x)}$ and $G(x) = -\Omega^2 f(x)h(x)$.

Let $f(x)$ be defined as

$$f(x) = \frac{\int^x -2h(\bar{x})\,\mathrm{d}\bar{x}}{h^2(x)}, \tag{3.6}$$

where $h(x)$ is any integrable real-valued function such that $f(x) > 0$. Define a pair of conjugate variables H and Θ in the spirit of Calogero and Levyraz by

$$H := \sqrt{f(x)}y, \qquad \Theta := \left(\int^x -2h(\bar{x})\,\mathrm{d}\bar{x}\right)^{\frac{1}{2}}, \tag{3.7}$$

where x and y are assumed to be canonical variables, having Poisson brackets $\{x, y\} = 1$. It is straightforward to verify that $\{\Theta, H\} = 1$ so that as before we define a Ω-modified Hamiltonian by

$$\widetilde{H} = \tfrac{1}{2}(H^2 + \Omega^2\Theta^2), \tag{3.8}$$

which gives rise to the following equations $\dot{\Theta} = H$ and $\dot{H} = -\Omega^2\Theta$ with solutions

$$H(t) = H(0)\cos(\Omega t) - \Theta(0)\Omega\sin(\Omega t) \tag{3.9}$$

$$\Theta(t) = \Theta(0)\cos(\Omega t) + \frac{H(0)}{\Omega}\sin(\Omega t). \tag{3.10}$$

It is easy to see that the evolution equations for x and y as determined by the Hamiltonian $\widetilde{H}$ are

$$\dot{x} = \frac{\partial\widetilde{H}}{\partial y} = f(x)y, \qquad \dot{y} = -\frac{\partial\widetilde{H}}{\partial x} = -\frac{f'(x)}{2}y^2 + \Omega^2 h(x), \tag{3.11}$$

where we have made explicit use of (3.8) and the definitions of H and Θ given in (3.7). Given a suitable function $h(x)$, one can in principle solve for x from $\Theta^2 = \int^x -2h(\bar{x})\,\mathrm{d}\bar{x}$, say $x(t) = K[\Theta^2(t)]$, and obtain $y(t) = \frac{H(t)}{[f(K(\Theta^2(t)))]^{\frac{1}{2}}}$ from the first equation in (3.7). As $H(t)$ and $\Theta(t)$ evolve periodically with time, it follows that x and y also evolve with the same period, namely $T = \frac{2\pi}{\Omega}$. The system of equations in (3.11) is equivalent to (3.5) by construction.

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